Time Series Analysis: TD1.

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Exercise 1.

1. Give an example of second order stationary time series such that $\gamma_X(h) = \sigma^2 > 0$ for any $h \geq 0$.
2. Give an example of strictly stationary times series which is not second order stationary.

Exercise 2. Consider $(X, Y) \sim \mathcal{N}_2(0, \Sigma)$ with

$$\Sigma = \begin{pmatrix} \sigma^2 & c \\ c & \sigma^2 \end{pmatrix}$$

1. Check that Σ is a symmetric Toeplitz matrix. Under which condition on c it is positive? Assume it in the sequel.
2. Provide the covariance and the correlation $\mathrm{Cov}\,(X, Y)$ and $\rho(X, Y)$.
3. Compute the conditional density $f_{X|Y=y}(x)$.
4. Deduce that the best prediction of X given Y is $\mathbb{E}[X \mid Y] = c/\sigma Y$. What is the associated quadratic risk $\mathbb{E}[(X - \mathbb{E}[X \mid Y])^2]$?
5. What is happening if $c = 1$?

Exercise 3. Consider a gaussian WN(1) (Z_t) and the time series (X_t) defined as

$$X_t = \begin{cases} Z_t, & \text{if } t \text{ is even,} \\ (Z_{t-1}^2 - 1)/\sqrt{2}, & \text{if } t \text{ is odd.} \end{cases}$$

We recall that $\mathbb{E}[Z_0^4] = 3$.

1. Check that $\mathbb{E}[Z_0^3] = 0$.
2. Show that $\mathbb{E}[X_t] = 0$ and $\mathrm{Var(X_t)} = 1$ for any $t \in \mathbb{Z}$.
3. Find the best predictions $\mathbb{E}[X_n \mid X_{n-1}, \dots, X_1]$ for n odd and n even.
4. Show that (X_t) is WN(1) but not SWN(1).
5. Deduce that the best linear prediction $\Pi_n(X_{n+1})$ equals 0 for any $n \geq 1$.
6. Compare the risk of the best predictions with the risk of the best linear prediction for n odd and n even.