Time Series Analysis: TD3.

Exercise 1. Let (X_t) be the MA(2) given by

$$X_t = Z_t + \theta Z_{t-2}$$

where (Z_t) is WN(0, 1).

1. Find the autocovariance and autocorrelation functions for this process when $\theta = 0.8$.
2. Compute the variance of the sample mean $\frac{X_1+X_2+X_3+X_4}{4}$ when $\theta = 0.8$.
3. Repeat (2) when $\theta = -0.8$ and compare your answer with the result obtained in (2).

Exercise taken from [1]

Exercise 2. Compute the autocorrelation (ACF) and the partial autocorrelation (PACF) functions of the AR(2) process

$$X_t = 0.8X_{t-2} + Z_t, \quad (Z_t) \sim \mathrm{WN}(0, \sigma^2).$$

Exercise taken from [1]

Exercise 3. Find a filter of the form $1 + \alpha L + \beta L^2 + \gamma L^3$ (i.e. find α, β and γ) that passes linear trends without distortion and that eliminates arbitrary seasonal components of period 2. We take $\bar{s} = \frac{1}{T}(s(1) + ... + s(T)) = 0$.
Exercise taken from [1]

Exercise 4. Show that the filter with coefficients $[a_{-2}, a_{-1}, a_0, a_1, a_2] = \frac{1}{9}[-1, 4, 3, 4, -1]$ passes third-degree polynomials and eliminates seasonal components with period 3. We take $\bar{s} = \frac{1}{T}(s(1) + ... + s(T)) = 0$.
Exercise taken from [1]

Exercise 5. This exercise is using the properties of the projection in order to get an efficient algorithm for determining the best linear prediction $\Pi_t(X_{t+1})$ and the associated risk R_t^L. Consider a WN(σ^2) (Z_t) and the MA(1) (X_t) defined as

$$X_t = Z_t + \theta Z_{t-1}\,, \qquad t \in \mathbb{Z}\,,$$

with $|\theta| < 1$.

1. Express the coefficients (φ_j) of the causal solution $X_t = \sum_{j=1}^{\infty} \varphi_j X_{t-j} + Z_t$ of the MA(1) model in term of θ.
2. Deduce $\Pi_\infty(X_{n+1})$ and the associated risk R_∞^L.

3. Show that $\Pi_n(X_{n+2}) = 0$ and $\mathbb{E}[X_{n+1}\Pi_{n-1}(X_n)] = 0$.
4. Deduce from the projection decomposition the recursive formula called

$$\Pi_n(X_{n+1}) = \frac{\sigma^2\theta}{R_n^L}(X_n - \Pi_{n-1}(X_n)), \qquad n \geq 1.$$

5. Deduce the recursive formula $R_{n+1}^L = \sigma^2(1+\theta^2) - \sigma^4\theta^2/R_n^L$ for $n \geq 1$ and the innovation algorithm that update $(\Pi_n(X_{n+1}), R_n^L)$ recursively.

[1] P. J. Brockwell and R. A. Davis. *Introduction to Time Series and Forecasting.* 2016